

Solitary waves of Bose-Einstein condensed atoms confined in finite rings

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Motivated by recent progress in trapping Bose-Einstein condensed atoms in toroidal potentials, we examine solitary-wave solutions of the nonlinear Schrödinger equation subject to periodic boundary conditions. When the circumference of the ring is much larger than the size of the wave, the density profile is well approximated by that of an infinite ring, however the density and the velocity of propagation cannot vanish simultaneously. When the size of the ring becomes comparable to the size of the wave, the density variation becomes sinusoidal and the velocity of propagation saturates to a constant value.

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I. INTRODUCTION

The problem of solitary-wave solutions of the one-dimensional nonlinear Schrödinger equation has been studied extensively, starting with the work of Zakharov and Shabat [1] and of Tsuzuki [2]. In addition, Lieb [3] has examined the problem of a Bose gas in strictly one dimension with periodic boundary conditions and a repulsive interaction, and he derived the excitation spectrum. As demonstrated in that study, the spectrum consists of the usual Bogoliubov branch as well as a second branch. This branch was later identified as corresponding to solitary-wave excitation [4, 5].

Bose-Einstein condensed atoms interacting via a contact potential are described by a nonlinear Schrödinger equation and thus offer a very suitable system for the study of solitary waves [6]. Solitary waves have been created in quasi-one-dimensional traps with a very weak harmonic potential in the long direction [7] and in pancake traps [8]. Recently, it also became possible to confine atoms in toroidal traps [9–13] in which persistent currents were observed [13].

Based on this experimental progress, it is natural to consider the possible creation of solitary waves in such toroidal traps. This is the subject of the present study. In the following, we examine the properties of solitary waves, including their density, propagation velocity, and phase (and thus the superfluid velocity), with periodic boundary conditions. We note that Carr *et al.* [14] have previously considered the special case of *static* periodic solitary-wave solutions for a fixed particle number and a length.

For simplicity we consider in the present study a very tight toroidal trap, neglecting the transverse degrees of freedom, and thus consider the purely one-dimensional motion of atoms on a ring. The main question to be considered here is the behavior of solitonic solutions as the length of the period L is varied for fixed N/L , where N is the number of particles. The relevant dimensionless parameter is the ratio of L and the coherence length, ξ , which we choose as that corresponding to the maximum

density. According to the value of L/ξ we distinguish between “large” rings, where $L/\xi \gg 1$, and “small” rings, where $L/\xi \ll 1$. In Lieb’s study [3], the limit $N \rightarrow \infty$ and $L \rightarrow \infty$ with N/L finite was considered. This corresponds to the limit of large rings in the present study. We note that the ratio $(\xi/L)^2$ also describes the ratio between the typical zero-point energy, $\hbar^2/(2ML^2)$, and the typical interaction energy, $\hbar^2/(2M\xi^2)$. Finally, we restrict ourselves to the case of repulsive effective interatomic interactions.

As we will show below, in the limit of large rings, the wave is localized within a length scale determined by the coherence length, and the density is constant over the rest of the system, in agreement with known results for an infinite system. On the other hand, depending on the phase difference at the edges of the system (which is either zero or some integer multiple of 2π), two different situations can arise. For zero phase difference, the minimum possible value of the density is zero, and the velocity of propagation varies between a lowest nonzero value, which scales as ξ/L and essentially corresponds to the superfluid “drift” velocity, and a maximum velocity which is set by the speed of sound. In the case of a nonzero phase difference, the velocity of propagation can be zero, but the corresponding density cannot vanish, scaling like $(\xi/L)^2$. Clearly, when $N \rightarrow \infty$ and $L \rightarrow \infty$ with N/L finite, we find agreement with known results.

In the opposite limit of small rings, the density becomes sinusoidal with a length scale given by L itself. Furthermore, the velocity of propagation saturates to a constant value, which scales like $1/L$, and the minimum density can become zero.

In what follows we first formulate our problem in Sec. II. Then, in Sec. III we give the general solution. Section IV examines certain limiting cases, starting from the linearization of the problem and then turning to the limits of large and small rings. In Sec. V we present a general result for the minimum value of the velocity of propagation of the wave, and finally we summarize our results and present our conclusions in Sec. VI.

II. FORMULATION OF THE PROBLEM

Assume that we have a ring of circumference L which contains N atoms of mass M . Within the mean-field approximation, the order parameter $\Phi(x, t)$ satisfies the equation (in the range $-L/2 \leq x \leq L/2$),

$$i\hbar \frac{\partial \Phi(x, t)}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2 \Phi(x, t)}{\partial x^2} + U_0 |\Phi(x, t)|^2 \Phi(x, t), \quad (1)$$

with $\int_{-L/2}^{L/2} |\Phi|^2 dx = N$ and $U_0 > 0$, because of the repulsive interatomic interactions that we consider. Assuming that $\Phi(x, t) = \Psi(z) e^{-i\mu t/\hbar}$, where μ is the chemical potential and $z = x - ut$,

$$-i\hbar u \frac{\partial \Psi}{\partial z} = -\frac{\hbar^2}{2M} \frac{\partial^2 \Psi}{\partial z^2} + (U_0 |\Psi|^2 - \mu) \Psi. \quad (2)$$

Here u is the velocity of propagation of the assumed traveling-wave solution. Setting $\Psi(z) = \sqrt{n(z)} e^{i\phi(z)}$, the superfluid velocity $v = (\hbar/M)\phi'(z)$ is given by the continuity equation

$$v(z) = \frac{\hbar}{M} \phi' = u + \frac{\hbar^3 C_1}{2M^2 n(z)}. \quad (3)$$

There is also an Euler equation for the density,

$$\frac{\hbar^2}{2M} [(\sqrt{n})']^2 + \mu n + \frac{\hbar^6 C_1^2}{8M^3 n} - \frac{U_0}{2} n^2 + \frac{M}{2} u^2 n = C_2, \quad (4)$$

where C_1 and C_2 are constants of integration.

The desired solution has a vanishing derivative at the points $z = 0$ and $\pm L/2$ where $n = n_{\min}$ and $n = n_{\max}$, respectively. This allows us to eliminate C_2 from Eq. (4) and to determine that

$$\frac{\hbar^6 C_1^2}{8M^3} = n_{\min} n_{\max} \left[\mu - \frac{1}{2} U_0 (n_{\min} + n_{\max}) + \frac{1}{2} M u^2 \right]. \quad (5)$$

This leads to

$$\frac{\hbar^2}{2M} (n')^2 = 2U_0 (n_{\max} - n)(n - n_{\min})(-n + n_0), \quad (6)$$

with $n_0 = (2\mu + Mu^2)/U_0 - n_{\min} - n_{\max}$. In the case of an infinite system, $\mu = n_{\max} U_0$ and $n_{\max} U_0 = Mu^2$, in which case Eq. (6) then reduces to the correct and known form of the differential equation.

III. THE GENERAL SOLUTION

The general solution of Eq. (6) which is periodic in the interval between $-L/2$ and $L/2$ is [14]

$$\begin{aligned} n(z) &= n_{\min} + \frac{1}{m} (n_{\max} - n_{\min}) \left[1 - \text{dn}^2 \left(\frac{2K(m)z}{L} |m \right) \right] \\ &= n_{\min} + (n_{\max} - n_{\min}) \text{sn}^2 \left(\frac{2K(m)z}{L} |m \right), \end{aligned} \quad (7)$$

where $\text{dn}(x|m)$ and $\text{sn}(x|m)$ are Jacobi elliptic functions and $K(m)$ is the elliptic integral of the first kind. Since $\text{sn}(x=0|m) = 0$ and $\text{sn}(x = \pm K(m)|m) = 1$, we see that $n(z=0) = n_{\min}$ and $n(z = \pm L/2) = n_{\max}$. In order for Eq. (7) to be the solution of the differential equation of Eq. (6),

$$K(m) = \frac{1}{2\sqrt{2}} \frac{L}{\xi} \left(\frac{1-\lambda}{m} \right)^{1/2}, \quad (8)$$

where ξ is the coherence length corresponding to a density n_{\max} , $\hbar^2/(2M\xi^2) = n_{\max} U_0$, and $\lambda = n_{\min}/n_{\max}$. Similarly, μ and C_1 are given as

$$\mu = -\frac{Mu^2}{2} + \frac{U_0}{2} \left[\left(2 - \frac{1}{m} \right) n_{\min} + \left(1 + \frac{1}{m} \right) n_{\max} \right], \quad (9)$$

and

$$\frac{\hbar^6 C_1^2}{8M^3} = \frac{U_0}{2} n_{\min} n_{\max} \left[\left(1 - \frac{1}{m} \right) n_{\min} + \frac{1}{m} n_{\max} \right]. \quad (10)$$

Since $K(m) \rightarrow \pi/2$ as $m \rightarrow 0$ and $K(m) \rightarrow \infty$ as $m \rightarrow 1$, m ranges between 0 and 1. Clearly the solution of Eq. (7) can be immediately extended to describe κ identical solitary waves on a ring of circumference κL .

The velocity of propagation, u , can then be determined by integrating the continuity equation, Eq. (3), over one period,

$$\begin{aligned} \int_{-L/2}^{L/2} v dz &= \frac{\hbar}{M} [\phi(L/2) - \phi(-L/2)] \\ &= uL + \frac{\hbar^3 C_1}{2M^2} \int_{-L/2}^{L/2} \frac{dz}{n(z)}. \end{aligned} \quad (11)$$

Assuming that $\phi(L/2) - \phi(-L/2) = 2\pi q$ with q an integer as a consequence of the periodic boundary conditions, we solve the above equation in terms of C_1 and combine it with Eq. (5) to obtain

$$\frac{u}{c} = \frac{2\sqrt{2}\pi q \xi}{L} \pm \frac{1}{L} \left[\lambda + \frac{1-\lambda}{m} \right]^{1/2} \int_{-L/2}^{L/2} \frac{\sqrt{n_{\min} n_{\max}}}{n(z)} dz. \quad (12)$$

Here c is the speed of sound of a homogeneous gas of density n_{\max} , $Mc^2 = n_{\max} U_0$. We remark from Eq. (11) that C_1 and u have opposite signs when $q = 0$.

While the parameterization of $n(z)$ that we adopt above is natural mathematically, it is of greater physical interest to explore solutions for fixed N , L , and U_0 as a function of, e.g., n_{\max} . To this end, it is convenient to pick a value of n_{\max} and m , and determine n_{\min} from Eq. (8). It is clear from its definition that n_{\min} must be greater than or equal to zero for physically meaningful solutions. The value of m can then be adjusted to satisfy the normalization condition on $n(z)$.

Equations (7), (8), and (12) are consistent with Bloch's theorem [15]. Because of the periodic boundary conditions imposed, the order parameter $\Psi_q(z)$ of the solitary wave corresponding to the branch with quantum number q is given by that obtained for $q = 0$ by exciting the center of mass motion, $\Psi_q(z) = e^{2\pi i q z/L} \Psi_0(z)$. These two solutions have the same density, but there is a difference in the velocity of propagation so that $[u(q) - u(q = 0)]/c = 2\sqrt{2}\pi q \xi/L$, in agreement with Eq. (12). Similarly, the energy spectrum consists of a periodic part plus an envelope function which results from the energy of the center of mass motion.

IV. LIMITING CASES

A. Linearization of the problem

When $n_{\min} \rightarrow n_{\max}$, one can obtain equations for the force and for continuity by linearizing the problem in small deviations of the density and the phase from the homogeneous solution $\Psi_0 = R_0 e^{i\phi_0}$,

$$\Psi = (R_0 - \delta R) e^{i(\phi_0 + \delta\phi)} \approx \sqrt{n_0} (1 - \alpha + i\delta\phi), \quad (13)$$

where $\alpha = \delta R/R_0$ and $\phi_0 = 0$. When the two are combined, we obtain

$$\frac{\hbar^2}{2M} \alpha''' + 2(Mu^2 - n_{\max} U_0) \alpha' = 0. \quad (14)$$

Thus, $\alpha' = C \sin(kz + \phi_0)$ with C and ϕ_0 constants. Periodicity requires that $kL = 2\pi l$ with l an integer. Here, we set $l = 1$ so that

$$\left(\frac{u}{c}\right)^2 = 1 + \frac{2\pi^2 \xi^2}{L^2}. \quad (15)$$

In the limit of “large” rings, i.e., $L \gg \xi$,

$$\frac{u}{c} \approx 1 + \frac{\pi^2 \xi^2}{L^2}. \quad (16)$$

In the opposite limit of “small” rings, $L \ll \xi$, then

$$\frac{u}{c} \approx \sqrt{2}\pi \frac{\xi}{L} \left(1 + \frac{L^2}{4\pi^2 \xi^2}\right). \quad (17)$$

We note that Eq. (15) can also be derived directly from Eq. (12) provided that one considers a fixed length L and then takes $n_{\min} \rightarrow n_{\max}$, which implies that $m \rightarrow 0$ in Eqs. (7) and (8).

B. Limit of large rings

We now consider the limiting forms of the solution appropriate for large and small rings. For large rings, $L \gg \xi$, and the general solution of Eq. (7) reduces to

$$n(z) = n_{\min} + (n_{\max} - n_{\min}) \tanh^2 \left(\frac{z}{\zeta} \right), \quad (18)$$

where $\zeta = \sqrt{2}\xi/(1 - n_{\min}/n_{\max})^{1/2}$. This is in agreement with the known result, plus corrections of order $e^{-L/\zeta}$. In this limit, $m \rightarrow 1$ with

$$m \approx 1 - 16 \exp \left[-\frac{L}{\sqrt{2}\xi} (1 - \lambda)^{1/2} \right] \quad (19)$$

and also $K(m) \approx \ln(4/\sqrt{1-m})$. From Eq. (12) we find that

$$\begin{aligned} \frac{u}{c} \approx & 2\sqrt{2}\pi q \frac{\xi}{L} + \sqrt{\frac{n_{\min}}{n_{\max}}} + \\ & + \frac{2\sqrt{2}\xi}{L} \tan^{-1} \left[\sqrt{\frac{n_{\max}}{n_{\min}}} - 1 \tanh \left(\frac{L}{2\zeta} \right) \right], \quad (20) \end{aligned}$$

which reduces to the familiar result $u/c = \sqrt{n_{\min}/n_{\max}}$ in the limit of an infinite ring.

Figure 1 shows u/c as a function of n_{\max} for the parameter choice $N/L = 1$, $L = 20$, $U_0 = 0.6$, and $q = 0$ (with $\hbar = M = 1$) so that $L/\xi > 20$. Exact results (dots) derived from Eq. (11), and values obtained using the approximation of Eq. (20) are given. The agreement is remarkably good except in the limit $n_{\max} \rightarrow n_{\min}$. In this limit, Eq. (20) yields $u/c = 1$, which does not contain the higher-order quadratic term seen in Eq. (16). In the opposite limit, where $n_{\min} \rightarrow 0$, Eq. (20) gives the exact result, as shown in Sec. V.

In the limit of small n_{\min} Eq. (20) reduces to

$$\frac{u}{c} \approx \frac{\sqrt{2}\pi\xi}{L} (1 + 2q) + \sqrt{\frac{n_{\min}}{n_{\max}}}. \quad (21)$$

Consider first the choice $q = 0$. In this case n_{\min} can be arbitrarily small in analogy with a “dark” solitary wave in the $L \rightarrow \infty$ limit. The velocity of propagation is bounded from below by ξ/L , $u_{\min}/c = \sqrt{2}\pi\xi/L$, which corresponds to the minimum value of u/c shown in Fig. 1. Equivalently,

$$\frac{\hbar}{M} \phi' = u_{\min} = \frac{\hbar}{M} \frac{\pi}{L} \quad (22)$$

except in the immediate vicinity of the solitary wave. This result is easily understood. As Eqs. (3) and (10) imply, when $n_{\min} = 0$, $C_1 = 0$ and the superfluid velocity is independent of position and equal to the velocity of propagation u . In the limit of a dark solitary wave, the phase $\phi(z)$ varies linearly, except in the limited region of small density, where the phase develops a discontinuity of $-\pi$. In order to satisfy the condition $\phi(L/2) = \phi(-L/2)$, $\phi(z)$ must have the small slope $\approx \pi/L$ over a large length $\approx L$. Thus, as Eq. (22) indicates, u cannot be zero.

In the case $q \neq 0$, u can vanish, but when u vanishes, n_{\min} is

$$\frac{n_{\min}}{n_{\max}} = \frac{2\pi^2 \xi^2}{L^2} (1 + 2q)^2. \quad (23)$$

This result is also easy to understand. Let us set $q = -1$. In this case Eq. (3) implies that $v(z) \propto 1/n(z)$. Away

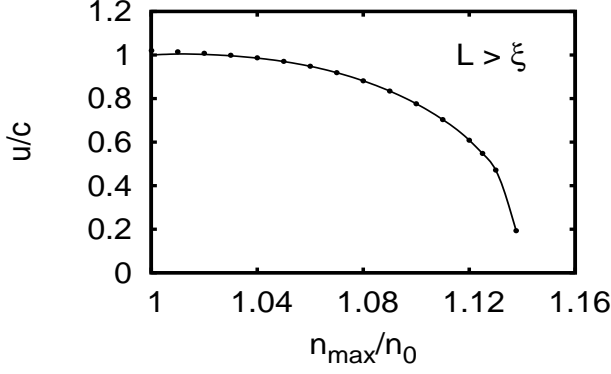


FIG. 1: The velocity of propagation u/c in “large” rings, as a function of n_{\max}/n_0 , where $n_0 = N/L$, with $n_0 = 1$, $L = 20$, $U_0 = 0.6$, and $q = 0$, with $\hbar = M = 1$. Here the dots give the exact result, Eq. (11), and the solid line gives the approximate result of Eq. (20). In this figure $L/\xi > 20$.

from the center of the wave, $v(z)$ is constant and equal to $(\hbar/M)(\pi/L)$ since the density is constant, but the phase $\phi(z)$ is equal to $\pi(z/L)$. The total accumulation of the phase over L is thus approximately π , while another π is needed in order to create the total phase difference of 2π . This is provided from the region around the wave where $v(z) \approx (\hbar/M)(\pi/L) n_{\max}/n(z)$, within a length scale of order $\xi\sqrt{n_{\min}/n_{\max}}$. This is the length scale of the dominant integration interval of $v(z)$ around its singularity. Since ϕ' is $\approx (\pi/L)(n_{\max}/n_{\min})$ in this regime, and in order for the integral of $\phi'(z)$ to be equal to π when we integrate it around a length scale of order $\xi\sqrt{n_{\min}/n_{\max}}$, we find that

$$\int \phi'(z) dz = \pi \approx \left(\frac{\pi}{L} \frac{n_{\max}}{n_{\min}} \right) \left(\xi \sqrt{\frac{n_{\min}}{n_{\max}}} \right). \quad (24)$$

This implies that $n_{\min}/n_{\max} \sim (\xi/L)^2$.

These results agree with those of Carr *et al.* [14]. In that study only static solitary waves were considered, and (as mentioned in Ref. [14]) this is not possible for zero phase difference. Their choice of $\xi/L = 1/25$ leads to $n_{\min}/n_{\max} \approx 0.032$ according to Eq. (23), which is in good agreement with the numerical results of Fig. 4 of this reference.

It should also be noted that Eq. (23) is valid for a single solitary-wave solution. For a two-solitary wave solution it is possible for them to be both static and dark (i.e., to have a node in the density).

The maximum possible value of u is achieved in the limit of sound waves $n_{\min} \rightarrow n_{\max}$ where

$$\frac{u_{\max}}{c} = 1 + 2\sqrt{2}\pi q \frac{\xi}{L} + \frac{\pi^2 \xi^2}{L^2}, \quad (25)$$

in agreement with Eq. (16). Finally, we note that the normalization condition takes the form $N/L = n_{\max} \left[1 - 2\sqrt{2}(\xi/L)(1-\lambda)^{1/2} \right]$ in this limit.

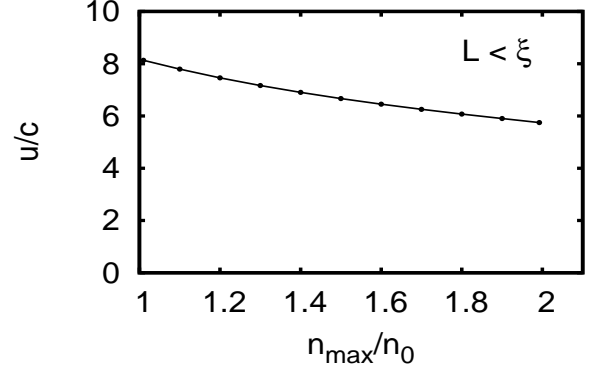


FIG. 2: The velocity of propagation u/c in “small” rings, as a function of n_{\max}/n_0 , where $n_0 = N/L$, with $n_0 = 1$, $L = 0.5$, $U_0 = 0.6$, and $q = 0$, with $\hbar = M = 1$. Here the dots give the exact result, Eq. (11), and the solid line gives the approximate result of Eq. (28). In this figure $L/\xi < 0.775$.

C. Limit of small rings

For small rings, $L \ll \xi$, the general solution of Eq. (7) reduces to

$$n(z) = n_{\min} + (n_{\max} - n_{\min}) \sin^2 \left(\frac{\pi z}{L} \right), \quad (26)$$

plus corrections of order m (i.e., of order $(L/\xi)^2$) since $m \approx (1-\lambda)(L^2/2\pi^2\xi^2)$ and $K(m) \approx (\pi/2)(1+m/4)$ in the limit $m \rightarrow 0$. Turning to the velocity of propagation of the wave, Eq. (12) implies that in the limit of small rings

$$\frac{u}{c} \approx \frac{\sqrt{2}\pi\xi}{L}(1+2q) + \frac{L}{2\sqrt{2}\pi\xi} \sqrt{\frac{n_{\min}}{n_{\max}}}. \quad (27)$$

One can identify the term proportional to $1+2q$ as the difference $(E_{q+1} - E_q)/\delta p$, where $E_q = \hbar^2 q^2 / (2MR^2)$ is the single-particle kinetic energy of a particle in a ring of radius R with angular momentum $\hbar q$, and $\delta p = \hbar/R$. This expression coincides with Eq. (17), found by linearization, when $q = 0$ and $n_{\min} \rightarrow n_{\max}$. Here, n_{\min} is not bounded from below as it was in the limit of large rings. Remarkably, u saturates to a constant value independent of n_{\min}/n_{\max} when $L/\xi \rightarrow 0$. This happens because

$$\begin{aligned} \frac{u}{c} &\approx 2\sqrt{2}\pi q \frac{\xi}{L} + \frac{1}{\pi} \left(\frac{1-\lambda}{m} \right)^{1/2} \int_{-\pi/2}^{\pi/2} \frac{\sqrt{\lambda} dw}{\lambda + (1-\lambda) \sin^2 w} \\ &= \frac{\sqrt{2}\pi\xi}{L}(1+2q) \end{aligned} \quad (28)$$

in this limit, since the integral is independent of λ . In other words, in the limit of small rings, the sinusoidal dependence of the density always gives rise to a phase difference of π for any value of the ratio n_{\min}/n_{\max} . Since the periodic boundary conditions require that the total

phase difference has to be $2\pi q$, the additional phase accumulation must be $(1+2q)\pi$. This fact gives a slope to the phase that is $\approx (1+2q)\pi/L$ (plus corrections of order $(L/\xi)^2$, and therefore $u \approx (1+2q)(\hbar/M)(\pi/L)$, in agreement with Eq. (28). Finally, the normalization condition is $N/L = (n_{\min} + n_{\max})/2$ in this limit.

Figure 2 shows u/c as a function of n_{\max} for the parameter choice $N/L = 1$, $L = 0.5$, $U_0 = 0.6$, and $q = 0$ (with $\hbar = M = 1$), so that $L/\xi < 0.775$. Exact results (dots) and values obtained using the approximation of Eq. (27) are given. The agreement is remarkably good in both limits $n_{\max} \rightarrow n_{\min}$, and $n_{\min} \rightarrow 0$, as Eq. (27) gives the exact result in both cases, as Eqs. (17) and (29) indicate.

V. A GENERAL RESULT

It is interesting to note that, in the limit of a dark solitary wave ($n_{\min} \rightarrow 0$), the velocity of propagation u of the wave is given by the same formula,

$$\frac{u_{\min}}{c} = \frac{\sqrt{2}\pi\xi}{L}(1+2q), \quad (29)$$

or

$$\phi' = \frac{\pi}{L}(1+2q), \quad (30)$$

in the limits of both small and large rings, as indicated by Eqs. (21) and (27). In fact, Eq. (29) gives the minimum possible value of u for *any* L : A change of variable transforms Eq. (12) into

$$\frac{u}{c} = \frac{2\sqrt{2}\pi q\xi}{L} + \frac{1}{2K(m)} \left[\lambda + \frac{1-\lambda}{m} \right]^{1/2} \times \int_{-1/\sqrt{\lambda}}^{1/\sqrt{\lambda}} \frac{dw}{[1 + (1-\lambda)w^2]\sqrt{1-\lambda w^2}\sqrt{1-\lambda m w^2}}. \quad (31)$$

The integral can be evaluated in the limit $\lambda \rightarrow 0$ by closing the integration contour with a semicircle in the upper half complex plane. The only contribution to the integral comes from the pole at $w = i$. In this way, one obtains Eq. (29) for any finite value of L .

In the limit where $n_{\min} \rightarrow 0$, the integral in Eq. (12) is dominated by the behavior of the density near $z = 0$ since $\text{sn}^2(x|m) \approx x^2$ for $x \rightarrow 0$. As a result, the integral is approximately proportional to $\int_{-\infty}^{\infty} dz/(\lambda + z^2) = \pi/\sqrt{\lambda}$. As a result, there is a contribution to ϕ' which is equal to π/L . For a nonzero q , there is an additional contribution of $2\pi q/L$ to ϕ' , from which Eqs. (29) and (30) follow.

VI. SUMMARY AND CONCLUSIONS

To summarize, solitary-wave solutions of the nonlinear Schrödinger equation framework show interesting fea-

tures if one imposes the periodic boundary conditions appropriate for the description of Bose-Einstein condensed atoms confined in tight toroidal traps.

In the limit of large rings, where the size of the wave is much smaller than the length of the ring, the density is exponentially localized on a length scale determined by the coherence length and is constant over the remainder of the ring. Depending on the phase difference at the edges of the period, either the minimum density can vanish (with the velocity of propagation being nonzero) or the velocity of propagation can vanish (with the minimum density being nonzero). However, both cannot be zero simultaneously.

In the limit of small rings, the density is sinusoidal within a length scale given by the circumference of the ring itself. The velocity of propagation saturates to a constant value, which is independent of n_{\min} . For this reason, the size of the ring must exceed some minimum size in order to create a static wave with $u = 0$ [14]. The minimum size can be determined from Eq. (31) for $q = -1$ and in the limit $m \rightarrow 0$ and $\lambda \rightarrow 1$ with $m/(1-\lambda) = 3$. Under these conditions, $u = 0$ and $L/\xi = \sqrt{6}\pi$, in agreement with Ref. [14].

In the limit of an infinite ring, the velocity of propagation of the solitary wave depends on the ratio n_{\min}/n_{\max} . In the opposite limit of small rings, the range of possible values of the velocity of propagation becomes narrower as L decreases. In the limit of very small L , u saturates to a constant value and becomes independent of n_{\min}/n_{\max} .

The general result that the minimum value of u is inversely proportional to L may be relatively easy to investigate experimentally.

Finally, we note that as opposed to the case of an infinite ring, which contains only the dimensionless quantity n_{\min}/n_{\max} , a second dimensionless quantity, L/ξ , appears in the present problem. In particular, in the limit of sound waves (where $1 - n_{\max}/n_{\max} \rightarrow 0$) and large rings (where $L/\xi \rightarrow \infty$), the product of these two terms appears, and the two limits “compete”. Different answers will result depending on the way in which these limits are taken. These different answers are not only of theoretical interest; they represent different physical situations determined by the way that an experiment is performed.

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